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To the boundary value problem of ordinary differential equations

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Abstract. A method for solving of a boundary value problem for ordinary differential equations with boundary conditions at phase and integral constraints is proposed. The base of the method is an immersion principle based on the general solution of the first order Fredholm integral equation which allows to reduce the original boundary value problem to the special problem of the optimal equation.

Keywords: boundary value problem of ordinary differential equations, the first order Fredholm integral equation, the principle of immersion, optimal control problem, optimization problem.

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1 Problem statement

We consider the following boundary value problem

$$\dot{x} = A(t)x + B(t)f(x,t) + \mu(t), \qquad t \in I = [t_0, t_1] \tag{1.1}$$

with boundary conditions

$$(x(t_0) = x_0,\ x(t_1) = x_1) \in S \subset R^{2n}, \tag{1.2}$$

with phase constraints

$$x(t) \in G(t) : G(t) = \{x \in R^n \mid \gamma(t) \le F(x,t) \le \delta(t),\ t \in I\}, \tag{1.3}$$

and integral constraints

$$3g_j(x) \le c_j, \qquad j = \overline{1, m_1}; \tag{1.4}$$

$$g_j(x) = c_j, \qquad j = \overline{m_1 + 1, m_2}; \tag{1.5}$$

$$g_j(x) = \int_{t_0}^{t_1} f_{0j}(x(t), t)\, dt, \qquad j = \overline{1, m_2}; \tag{1.6}$$

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Here $A(t)$, $B(t)$ are prescribed matrices with piecewise continuous elements of $n \times n$, $n \times m$ order, respectively, $\mu(t)$, $t \in I$ is given n-dimensional vector-function with piecewise continuous elements, m-dimensional vector-function $f(x,t)$ is defined and continuous in the variables $(x,t) = R^n \times I$ and satisfies the following conditions:

$$|f(x,t) - f(y,t)| \le l|x-y|, \qquad \forall (x,t), \qquad (y,t) \in R^n \times I, \qquad l = \text{const} > 0,$$
$$|f(x,t)| \le c_0|x| + c_1(t), \qquad c_0 = \text{const} \ge 0, \qquad c_1(t) \in L_1(I, R^1),$$

S is a convex closed set. Function $F(x,t) = (F_1(x,t), \dots, F_r(x,t))$, $t \in I$ is an r-dimensional vector-function which is continuous in arguments, $\gamma(t) = (\gamma_1(t), \dots, \gamma_r(t))$ and $\delta(t) = (\delta_1(t), \dots, \delta_r(t))$, $t \in I$ are prescribed continuous functions.

The values c_j, $j = \overline{1, m_2}$ are prescribed constants, $f_{0j}(x,t)$, $j = \overline{1, m_2}$ are given continuous functions satisfying to the conditions

$$|f_{0j}(x,t) - f_{0j}(y,t)| \le l_j|x-y|, \qquad \forall (x,t), \qquad (y,t) \in R^n \times I, \qquad j = \overline{1, m_2};$$
$$|f_{0j}(x,t)| \le c_{0j}|x| + c_{1j}(t), \qquad c_{0j} = \text{const}, \qquad c_{1j} \in L_1(I, R^1), \qquad j = \overline{1, m_2}.$$

Note, that: 1) if $A(t) \equiv 0$, $m = n$, $B(t) = I_n$, then the equation (1.1) can be written as

$$\dot{x} = f(x,t) + \mu(t) = \overline{f}(x,t), \qquad t \in I. \tag{1.7}$$

Therefore, the results obtained below remain valid for the equation (1.7) at conditions (1.2)–(1.6);

2) if $f(x,t) = x + \mu_1(t)$ (or $f(x,t) = C(t)x + \mu_1(t)$), then the equation (1.1) can be written in form

$$\dot{x} = A(t)x + B(t)x + B(t)\mu_1(t) + \mu(t) = \overline{A}(t)x + \overline{\mu}(t), \qquad t \in I, \tag{1.8}$$

where $\overline{A}(t) = A(t) + B(t)$, $\overline{\mu}(t) = B(t)\mu_1(t) + \mu(t)$. It follows that the equation (1.8) is a particular case of equation (1.1).

The following problems are stated.

Problem 1. To find necessary and sufficient conditions for the existence of solutions of boundary value problem (1.1)–(1.6).

Problem 2. To construct a solution of boundary value problem (1.1)–(1.6).

As it follows from the problem statement, it is necessary to prove the existence of the pair $(x_0, x_1) \in S$ such that the solution of (1.1) proceeded from the point x_0 at the time t_0 passes through the point x_1 at the time t_1, along with the solution of the system (1.1) for each time the phase constraint is satisfied (1.3), and integrals (1.6) satisfy (1.4), (1.5). In particular, the set S is defined by the relation

$$S = \left\{ (x_0, x_1) \in R^{2n} \mid H_j(x_0, x_1) \le 0,\ j = \overline{1,p};\ \langle a_j, x_0 \rangle + \langle b_j, x_1 \rangle - d_j = 0,\ j = \overline{p+1, s} \right\},$$

where $H_j(x_0, x_1)$, $j = \overline{1,p}$ are convex functions in the variables (x_0, x_1), $x_0 = x(t_0)$, $x_1 = x(t_1)$, $a_j \in R^n$, $b_j \in R^n$, $d_j \in R^1$, $j = \overline{p+1, s}$ are given vectors and numbers, $\langle \cdot, \cdot \rangle$ is the scalar product.

In many cases, in practice the process under study is described by the equation of the form (1.1) in the phase space of the system defined by the phase constraint of the form (1.3). Outside this domain the process is described by completely different equations or the process under investigation does not exist. In particular, such phenomena take place in the research of dynamics of nuclear and chemical reactors (outside the domain (1.3) reactors do not exist.)

Integral constraints of the form (1.4) characterize the total load experienced by the elements and nodes in the system (for example, total overload of cosmonauts), which should not exceed the specified values and equations of the form (1.5) correspond to the total limits for the system (for example, fuel consumption is equal to a predetermined value).

The essence of the method consists in the fact that at the first stage of research by transformation and introducing a fictitious control the initial problem is immersed in the control problem. Further, the existence of solutions of the original problem and the construction of its solution is carried out by solving the problem of optimal control of a special kind. With this approach, the necessary and sufficient conditions for the existence of the solution of the boundary value problem (1.1)–(1.6) can be obtained from the condition to achieve the lower bound of the functional on a given set, and the solution of the original boundary problem is the limit points of minimizing sequences.

2 Transformation

We assume that $f_0(x,t) = (f_{01}(x,t), \dots, f_{0m_2}(x,t))$, where

$$f_0(x,t) = C(t)x + \overline{f_0}(x,t), \qquad t \in I, \tag{2.1}$$

$C(t)$, $t \in I$ is known matrix of $m_2 \times n$ order with piecewise continuous elements, $\overline{f}_0(x,t) = (\overline{f}_{01}(x,t), \dots, \overline{f}_{0m}(x,t))$. If the j-th row of the matrix $C(t)$ is zero, then $f_{0j}(x,t) = \overline{f}_{0j}(x,t)$. Thus, without loss of generality, we can assume the function $f_0(x,t)$ is defined by (2.1). By introducing additional variables $d = (d_1, \dots, d_{m_1}) \in R^{m_1}$, $d \geq 0$, the relations (1.4), (1.6) can be represented as

$$g_j(x) = \int_{t_0}^{t_1} f_{0j}(x(t),t)\,dt = c_j - d_j, \qquad j = \overline{1, m_1},$$

where

$$d \in \Gamma = \{d \in R^{m_1} \mid d \geq 0\}.$$

Let the vector $\bar{c} = (\bar{c}_1, \dots, \bar{c}_{m_2})$, where $\bar{c}_j = c_j - d_j$, $j = \overline{1, m_1}$, $\bar{c}_j = c_j$, $j = \overline{m_1 + 1, m_2}$. We introduce vector-function $\eta(t) = (\eta_1(t), \dots, \eta_{m_2}(t))$, $t \in I$, where

$$\eta(t) = \int_{t_0}^{t} f_0(x(\tau),\tau)\,d\tau, \qquad t \in [t_0, t_1].$$

Then

$$\dot{\eta} = f_0(x(t),t) = C(t)x + \overline{f}_0(x,t), \qquad t \in I$$
$$\eta(t_0) = 0, \qquad \eta(t_1) = \bar{c}, \qquad d \in \Gamma.$$

Now the initial boundary value problem (1.1)–(1.6) can be written as

$$\dot{\xi} = A_1(t)\xi + B_1(t)f(P\xi,t) + B_2\overline{f}_0(P\xi,t) + B_3\mu(t), \qquad t \in I, \tag{2.2}$$

$$\xi(t_0) = \xi_0 = (x_0, O_{m_2}), \qquad \xi(t_1) = \xi_1 = (x_1, \bar{c}), \tag{2.3}$$

$$(x_0, x_1) \in S, \;\; d \in \Gamma, \qquad P\xi(t) \in G(t), \;\; t \in I, \tag{2.4}$$

where

$$\xi(t) = \begin{pmatrix} x(t) \\ \eta(t) \end{pmatrix}, \qquad A_1(t) = \begin{pmatrix} A(t) & O_{n,m_2} \\ C(t) & O_{m_2,m_2} \end{pmatrix}, \qquad B_1(t) = \begin{pmatrix} B(t) \\ O_{m_2,m} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} I_n \\ O_{m_2,n} \end{pmatrix}, \qquad B_3 = \begin{pmatrix} O_{n,m_2} \\ I_{m_2} \end{pmatrix}, \qquad P = \begin{pmatrix} I_n, & O_{n,m_2} \end{pmatrix}, \qquad P\xi = x,$$

$O_{j,k}$ is matrix of $j \times k$ order with zero elements, $O_q \in R^q$ is vector $q \times 1$ with zero elements, $\xi = (\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_{n+m_2})$.

3 Integral equation

The basis of the proposed method of solving problems 1 and 2 are the following theorems about the properties of solutions of the first order Fredholm integral equation:

$$Ku = \int_{t_0}^{t_1} K(t_0, t)u(t)\,dt = a, \tag{3.1}$$

where $K(t_0, t) = \|K_{ij}(t_0, t)\|$, $i = \overline{1, n}$, $j = \overline{1, m}$ is known matrix of $n \times m$ order with piecewise continuous elements in t at fixed t_0, $u(\cdot) \in L_2[I, R^m]$ is the source function, $I = [t_0, t_1]$, $a \in R^n$ is given n-dimensional vector.

Theorem 3.1. *Integral equation (3.1) for any fixed $a \in R^n$ has a solution if and only if the matrix*

$$C(t_0, t_1) = \int_{t_0}^{t_1} K(t_0, t)K^*(t_0, t)\,dt, \tag{3.2}$$

$n \times n$ order is positive definite, where "" is a sign of transposition.*

Theorem 3.2. *Let the matrix $C(t_0, t_1)$ be positive definite. Then the general solution of the integral equation (3.1) has the form*

$$u(t) = K^*(t_0, t)C^{-1}(t_0, t_1)a + v(t) - K^*(t_0, t)C^{-1}(t_0, t_1)\int_{t_0}^{t_1} K(t_0, t)v(t)\,dt, \qquad t \in I, \tag{3.3}$$

where $v(\cdot) \in L_2(I, R^m)$ is an arbitrary function, $a \in R^n$ is an arbitrary vector.

Proofs of Theorems 3.1 and 3.2 are given in [2,3]. Application of Theorems 3.1 and 3.2 to solve the controllability and optimal control problem is presented in [4–7].

4 Immersion principle

Along with the differential equation (2.2) with boundary conditions (2.3) we consider the linear control system

$$\dot{y} = A_1(t)y + B_1(t)w_1(t) + B_2(t)w_2(t) + \mu_2(t), \qquad t \in I, \tag{4.1}$$

$$y(t_0) = \xi_0 = (x_0, O_{m_2}), \qquad y(t_1) = \xi_1 = (x_1, \bar{c}), \tag{4.2}$$

$$(x_0, x_1) \in S, \qquad d \in \Gamma, \qquad w_1(\cdot) \in L_2(I, R^m), \qquad w_2(\cdot) \in L_2(I, R^{m_2}), \tag{4.3}$$

where $\mu_2(t) = B_3\mu(t)$, $t \in I$.

Let the matrix $\overline{B}(t) = (B_1(t), B_2(t))$ of $(n + m_2) \times (m_2 + m)$ order, and the vector-function

$$w(t)=\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} \in L_2(I, R^{m+m_2}).$$

It is easy to see that the control $w(\cdot) \in L_2(I, R^{m+m_2})$ which transfers the trajectory of system (4.1) from any initial state ξ_0 to any desired state ξ_1 is a solution of the integral equation

$$\int_{t_0}^{t_1} \Phi(t_0, t)\overline{B}(t)w(t)\,dt = a, \tag{4.4}$$

where $\Phi(t, \tau) = \theta(t)\theta^{-1}(\tau)$, $\theta(t)$ is the fundamental matrix of solutions of the linear homogeneous system $\dot{\omega} = A_1(t)\omega$, vector

$$a = a(\xi_0, \xi_1) = \Phi(t_0, t_1)[\xi_1 - \Phi(t_1, t_0)\xi_0] - \int_{t_0}^{t_1} \Phi(t_0, t)\mu_2(t)\,dt.$$

As follows from (3.1), (4.4), the matrix $K(t_0, t) = (t_0, t)\overline{B}(t)$. We introduce the following notations

$$\lambda_1(t, \xi_0, \xi_1) = T_1(t)\xi_0 + T_2(t)\xi_1 + \mu_3(t) = E(t)a, \qquad t \in I,$$

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)\overline{B}(t)\overline{B}^*(t)\Phi^*(t_0, t)dt, \qquad W(t_0, t) = \int_{t_0}^{t} \Phi(t_0, \tau)\overline{B}(\tau)\overline{B}^*(\tau)\Phi^*(t_0, \tau)d\tau,$$

$$W(t, t_1) = W(t_0, t_1) - W(t_0, t), \qquad E(t) = \overline{B}^*(t)\Phi^*(t_0, t)W^{-1}(t_0, t_1),$$

$$\mu_3(t) = -E(t)\int_{t_0}^{t_1} \Phi(t_0, t)\mu_2(t)dt, \qquad \lambda_2(t, \xi_0, \xi_1) = E_1(t)\xi_0 + E_2(t)\xi_1 + \mu_4(t),$$

$$E_1(t) = \Phi(t, t_0)W(t, t_1)W^{-1}(t_0, t_1), \qquad E_2(t) = \Phi(t, t_0)W(t_0, t)W^{-1}(t_0, t_1)\Phi(t_0, t_1),$$

$$\mu_4(t) = \Phi(t, t_0)\int_{t_0}^{t} \Phi(t_0, \tau)\mu_2(\tau)d\tau - E_2(t)\int_{t_0}^{t_1} \Phi(t_1, t)\mu_2(t)dt,$$

$$N_1(t) = -E(t)\Phi(t_0, t_1), \qquad N_2(t) = -E_2(t), \qquad t \in I.$$

Theorem 4.1. *Let the matrix $W(t_0, t_1) > 0$. The control $w(\cdot) \in L_2(I, R^{m+m_2})$ transfers the trajectory of system (4.1) from any initial point $\xi_0 \in R^{n+m_2}$ to any finite state $\xi_1 \in R^{n+m_2}$ if and only if*

$$w(t) \in W = \Big\{ w(\cdot) \in L_2(I, R^{m+m_2}) \mid w(t) = v(t) + \lambda_1(t, \xi_0, \xi_1) + N_1(t)z(t_1, v), \\ t \in I,\ \forall v(\cdot) \in L_2(I, R^{m+m_2}) \Big\}, \tag{4.5}$$

where function $z(t) = z(t, v)$, $t \in I$ is a solution of the differential equation

$$\dot{z} = A_1(t)z + \overline{B}(t)v(t), \qquad z(t_0) = 0, \qquad t \in I, \qquad v(\cdot) \in L_2(I, R^{m+m_2}). \tag{4.6}$$

Solution of the differential equation (4.1) corresponding to the control $w(t) \in W$ is defined by the formula

$$y(t) = z(t) + \lambda_2(t, \xi_0, \xi_1) + N_2(t)z(t_1, v), \qquad t \in I. \tag{4.7}$$

Proof. As follows from Theorem 3.1, the matrix $W(t_0,t_1) = C(t_0,t_1) > 0$, where $K(t_0,t) = \Phi(t_0,t)\overline{B}(t)$. Now the relation (3.3) is written in the form (4.5). Solution of the system (4.1) corresponding to the control (4.5) is defined by (4.7), where $z(t) = z(t,v)$, $t \in I$ is solution of the differential equation (4.6). The theorem is proved. □

Lemma 4.2. *Let the matrix $W(t_0,t_1) > 0$. Then the boundary value problem (1.1)–(1.6) (or (2.2)–(2.4)) is equivalent to the following problem*

$$w(t) = (w_1(t), w_2(t)) \in W, \qquad w_1(t) = f(Py(t),t), \qquad w_2(t) = \overline{f}_0(Py(t),t), \tag{4.8}$$

$$\dot{z} = A_1(t)z + B_1(t)v_1(t) + B_2(t)v_2(t), \qquad z(t_0) = 0, \qquad t \in I, \tag{4.9}$$

$$v(t) = (v_1(t), v_2(t)), \qquad v_1(\cdot) \in L_2(I,R^m), \qquad v_2(\cdot) \in L_2(I,R^{m_2}), \tag{4.10}$$

$$(x_0,x_1) \in S, \qquad d \in \Gamma, \qquad Py(t) \in G(t), \qquad t \in I, \tag{4.11}$$

where $v(\cdot) = (v_1(\cdot), v_2(\cdot)) \in L_2(I,R^{m+m_2})$ is an arbitrary function, $y(t)$, $t \in I$ is determined by the formula (4.7).

Proof. At relations (4.8)–(4.11) function

$$y(t) = \xi(t), \qquad t \in I, \qquad Py(t) = P\xi(t) \in G(t), \qquad t \in I, \qquad w(t) = (w_1(t), w_2(t)) \in W.$$

The lemma is proved. □

We consider the following optimization problem: minimize the functional

$$\begin{aligned} &J(v_1,v_2,p,d,x_0,x_1) \\ &\quad= \int_{t_0}^{t_1} \left[|w_1(t) - f(Py(t),t)|^2 + |w_2(t) - \overline{f}_0(Py(t),t)|^2 + |p(t) - F(Py(t),t)|^2\right] dt \\ &\quad= \int_{t_0}^{t_1} F_0(t,v_1(t),v_2(t),p(t),d,x_0,x_1,z(t),z(t_1))\, dt \to \inf \end{aligned} \tag{4.12}$$

at conditions

$$\dot{z} = A_1(t)z + B_1(t)v_1(t) + B_2(t)v_2(t), \qquad z(t_0) = 0, \qquad t \in I, \tag{4.13}$$

$$v_1(\cdot) \in L_2(I,R^m), \qquad v_2(\cdot) \in L_2(I,R^{m_2}), \qquad (x_0,x_1) \in S, \qquad d \in \Gamma, \tag{4.14}$$

$$p(t) \in P(t) = \{p(\cdot) \in L_2(I,R^r) \mid \gamma(t) \le p(t) \le \delta(t),\ t \in I\}, \tag{4.15}$$

where

$$w_1(t) = v_1(t) + \lambda_{11}(t,\xi_0,\xi_1) + N_{11}(t)z(t_1,v), \qquad t \in I,$$

$$w_2(t) = v_2(t) + \lambda_{12}(t,\xi_0,\xi_1) + N_{12}(t)z(t_1,v), \qquad t \in I,$$

$$N_1(t) = \begin{pmatrix} N_{11}(t) \\ N_{12}(t) \end{pmatrix}, \qquad \lambda_1(t,\xi_0,\xi_1) = \begin{pmatrix} \lambda_{11}(t,\xi_0,\xi_1) \\ \lambda_{12}(t,\xi_0,\xi_1) \end{pmatrix}.$$

We denote

$$\begin{aligned} X &= L_2(I,R^{m+m_2}) \times P(t) \times \Gamma \times S \subset H \\ &= L_2(I,R^m) \times L_2(I,R^{m_2}) \times L_2(I,R^r) \times R^{m_1} \times R^n \times R^n, \end{aligned}$$

$$J_* = \inf_{\theta \in X} J(\theta), \qquad \theta = (v_1,v_2,p,d,x_0,x_1) \in X, \qquad X_* = \left\{\theta_* \in \frac{X}{J(\theta_*)} = 0\right\}.$$

Theorem 4.3. *Let the matrix $W(t_0, t_1) > 0$, $X_* \neq \emptyset$. In order for the boundary value problem (1.1)–(1.6) to have a solution, it is necessary and sufficient that the value $J(\theta_*) = 0 = J_*$, where $\theta_* = (v_1^*, v_2^*, p_*, d_*, x_0^*, x_1^*) \in X$ is optimal control for the problem (4.12)–(4.15).*

If $J_* = J(\theta_*) = 0$, then the function

$$x_*(t) = z(t, v_1^*, v_2^*) + \lambda_2(t, d_*, x_0^*, x_1^*) + N_2(t) z(t_1, v_1^*, v_2^*), \qquad t \in I$$

is a solution of the boundary value problem (1.1)–(1.6). If $J_* > 0$, then the boundary value problem (1.1)–(1.6) has no solution.

Proof. Necessity. Let the boundary value problem (1.1)–(1.6) have a solution. Then it follows from Lemma 4.2, that the values $w_1^*(t) = f(Py_*(t), t)$, $w_2^*(t) = \overline{f}_0(Py_*(t), t)$, where $w_*(t) = (w_1^*(t), w_2^*(t)) \in W$, $y(t)$, $t \in I$ is defined by formula (4.7), $\xi_0^* = (x_0^*, O_{m_2})$, $\xi_1^* = (x_1^*, \overline{c}_*)$, $\overline{c}_* = (c_j{}^* - d_j^*, j = \overline{1, m_1}, c_j, j = \overline{m_1 + 1, m_2})$. Inclusion $y_*(t) \in G(t)$, $t \in I$ is equivalent to $p_*(t) = F(y_*(t), t)$, where $\gamma(t) \leq p_*(t) = F(y_*(t), t) \leq \delta(t)$, $t \in I$. Consequently, the value $J(\theta_*) = 0$. Necessity is proved.

Sufficiency. Let $J(\theta_*) = 0$. This is possible if and only if $w_1^*(t) = f(Py_*(t), t)$, $w_2^*(t) = \overline{f}_0(Py_*(t), t)$, $p_*(t) = F(y_*(t), t)$, $(x_0^*, x_1^*) \in S$, $d_* \in \Gamma$, $v_1^*(\cdot) \in L_2(I, R^m)$, $v_2^*(\cdot) \in L_2(I, R^{m_2})$. Sufficiency is proved. The theorem is proved. □

The transition from the boundary value problem (1.1)–(1.6) to the problem (4.12)–(4.15) is called the principle of immersion.

5 Optimization problem

We consider the solution of the optimization problem (4.12)–(4.15). Note, that the function

$$\begin{aligned} F_0(t,\ v_1,\ v_2,\ p,\ d,\ x_0,\ x_1,\ z,\ \overline{z}) &= |w_1 - f(Py, t)|^2 + |w_2 - \overline{f_0}(Py, t)|^2 + |p - F(Py, t)|^2 \\ &= F_0(t, \theta, z, \overline{z}) = F_0(t, q), \qquad q = (\theta, z, \overline{z}), \end{aligned}$$

where

$$\begin{aligned} w_1 &= v_1 + \lambda_{11}(t, x_0, x_1, d) + N_{11}(t)\overline{z}, & \overline{z} &= z(t_1, v_1, v_2), \\ w_2 &= v_2 + \lambda_{12}(t, x_0, x_1, d) + N_{12}(t)\overline{z}, & y &= z + \lambda_2(t, x_0, x_1, d) + N_2(t)\overline{z}, \\ P &= (I_n, O_{nm_2}), & Py &= x. \end{aligned}$$

Theorem 5.1. *Let the matrix be $W(t_0, t_1) > 0$, the function $F_0(t, q)$ is defined and continuously differentiable in $q = (\theta, z, \overline{z})$, and the following conditions hold:*

$$|F_{0z}(t, \theta + \Delta\theta, z + \Delta z, \overline{z} + \Delta\overline{z}) - F_{0z}(t, \theta, z, \overline{z})| \leq L(|\Delta z| + |\Delta\overline{z}| + |\Delta\theta|),$$

$$|F_{0\overline{z}}(t, \theta + \Delta\theta, z + \Delta z, \overline{z} + \Delta\overline{z}) - F_{0\overline{z}}(t, \theta, z, \overline{z})| \leq L(|\Delta z| + |\Delta\overline{z}| + |\Delta\theta|),$$

$$|F_{0\theta}(t, \theta + \Delta\theta, z + \Delta z, \overline{z} + \Delta\overline{z}) - F_{0\theta}(t, \theta, z, \overline{z})| \leq L(|\Delta z| + |\Delta\overline{z}| + |\Delta\theta|),$$

$$\forall \theta \in R^{m+m_2+r+m_1+n+n}, \qquad \forall z \in R^{n+m_2}, \qquad \forall \overline{z} \in R^{n+m_2}.$$

Then the functional (4.12) at conditions (4.13)–(4.15) is continuous and differentiable by Fréchet in any point $\theta \in X$, and

$$J'(\theta) = (J_1'(\theta), J_2'(\theta), J_3'(\theta), J_4'(\theta), J_5'(\theta), J_6'(\theta)) \in H,$$

where

$$J_1'(\theta) = \frac{\partial F_0(t,q)}{\partial v_1} - B_1^*(t)\psi(t), \qquad J_2'(\theta) = \frac{\partial F_0(t,q)}{\partial v_1} - B_2^*(t)\psi(t),$$
$$J_3'(\theta) = \frac{\partial F_0(t,q)}{\partial p}, \qquad J_4'(\theta) = \int_{t_0}^{t_1} \frac{\partial F_0(t,q)}{\partial d}\,dt, \tag{5.1}$$
$$J_5'(\theta) = \int_{t_0}^{t_1} \frac{\partial F_0(t,q)}{\partial x_0}\,dt, \qquad J_6'(\theta) = \int_{t_0}^{t_1} \frac{\partial F_0(t,q)}{\partial x_1}\,dt,$$

$q = (\theta, z(t), z(t,v))$, function $z(t)$, $t \in I$ is solution of differential equation (4.13) at $v_1 = v_1(\cdot) \in L_2(I,R^m)$, $v_2 = v_2(\cdot) \in L_2(I,R^{m_2})$, and function $\psi(t)$, $t \in I$ is solution of the adjoint system

$$\dot{\psi} = \frac{\partial F_0(t,q(t))}{\partial z} - A_1^*(t)\psi, \psi(t_1) = -\int_{t_0}^{t_1} \frac{\partial F_0(t,q(t))}{\partial z(t_1)}\,dt. \tag{5.2}$$

In addition, the gradient $J'(\theta)$, $\theta \in X$ satisfies to Lipschitz condition

$$\|J'(\theta_1) - J'\theta_2)\| \le K\|\theta_1 - \theta_2\|, \qquad \forall \theta_1, \theta_2 \in X, \tag{5.3}$$

where $K > 0$ is Lipschitz constant.

Proof. Let θ, $\theta + \Delta\theta \in X$, where $\Delta\theta = (\Delta v_1, \Delta v_2, \Delta p, \Delta d, \Delta x_0, \Delta x_1)$. It can be shown that $\Delta \dot{z} = A_1(t)\Delta z + B_1(t)\Delta v_1 + B_2(t)\Delta v_2$, increment of the functional

$$\begin{aligned}\Delta J &= J(\theta + \Delta\theta) - J(\theta)\\ &= \langle J_1'(\theta), \Delta v_1\rangle_{L_2} + \langle J_2'(\theta), \Delta v_2\rangle_{L_2} + + \langle J_3'(\theta), \Delta p\rangle_{L_2} + \langle J_4'(\theta), \Delta d\rangle_{R^{m_1}}\\ &\quad + \langle J_5'(\theta), \Delta x_0\rangle_{R^n} + \langle J_6'(\theta), \Delta x_1\rangle_{R^n} + R_1 + R_2 + R_3 + R_4 + R_5 + R_6,\end{aligned}$$

where $|R_1 + R_2 + R_3 + R_4 + R_5 + R_6| \le c_* \|\Delta\theta\|_X^2$, $c_* = \text{const} > 0$, $\frac{|\sum_{i=1}^6 R_i|}{\|\Delta\theta\|_X} \to 0$ at $\|\Delta\theta\|_X \to 0$. From this, the statement of the theorem follows. The theorem is proved. □

Using the relations (5.1)–(5.3) we construct a sequence $\{\theta_n\} = \{v_m(t), v_{2n}(t), p_n(t)\}$ by the following algorithm:

$$\begin{aligned} v_{1n+1} &= v_{1n} - \alpha_n J_1'(\theta_n), & v_{2n+1} &= v_{2n} - \alpha_n J_2'(\theta_n),\\ p_{n+1} &= P_p[p_n - \alpha_n J_3'(\theta_n)], & d_{n+1} &= P_\Gamma[d_n - \alpha_n J_4'(\theta_n)],\\ x_{0n+1} &= P_S[x_{0n} - \alpha_n J_5'(\theta_n)] & &\\ x_{1n+1} &= P_S[x_{1n} - \alpha_n J_6'(\theta_n)], & n &= 0,1,2,\ldots, \end{aligned} \tag{5.4}$$

where $0 < \alpha_n = \frac{2}{K+2\varepsilon}$, $\varepsilon > 0$, $K > 0$ is Lipschitz constant of (5.3). We introduce the following sets

$$\Lambda_0 = \{\theta \in X \mid J(\theta) \le J(\theta_0)\}, \qquad X_{**} = \Big\{\theta_{**} \in X \mid J(\theta_{**}) = \inf_{\theta \in X} J(\theta)\Big\}.$$

Theorem 5.2. *Let the conditions of Theorem 5.1 be satisfied, the functional $J(\theta)$, $\theta \in X$ be bounded from below, the sequence $\{\theta_n\} \subset X$ be defined by (5.4). Then:*

$$1)\quad J(\theta_n) - J(\theta_{n+1}) \ge \varepsilon\|\theta_n - \theta_{n+1}\|^2, \qquad n = 0,1,2,\ldots; \tag{5.5}$$

$$2)\quad \lim_{n\to\infty} \|\theta_n - \theta_{n+1}\| = 0, \qquad n = 0,1,2,\ldots. \tag{5.6}$$

Proof. Since θ_{n+1} is the projection of the point $\theta_n - \alpha_n J'(\theta_n)$, then

$$\langle \theta_{n+1} - \theta_n + \alpha_n J'(\theta_n), \theta_n - \theta_{n+1} \rangle_H \geq 0, \qquad \forall\, \theta \in X.$$

Then with taking into account $J(\theta) \in C^{1,1}(X)$ we get

$$J(\theta_n) - J(\theta_{n+1}) \geq \left(\frac{1}{\alpha_n} - \frac{K}{2} \right) \|\theta_n - \theta_{n+1}\|^2 \geq \varepsilon \|\theta_n - \theta_{n+1}\|^2.$$

Consequently, the numerical sequence $\{J(\theta_n)\}$ is strictly decreasing, and the inequality (5.5) is valid. Equality (5.6) follows from the boundedness below of functional $J(\theta)$, $\theta \in X$. The theorem is proved. □

Theorem 5.3. *Let the conditions of Theorem 5.1 hold, the set Λ_0 be bounded, $J(\theta)$, $\theta \in X$ be convex functional. Then the following statements hold.*

1) The set Λ_0 is weakly bicompact, $X_{**} \neq 0$.

2) The sequence $\{\theta_n\}$ is minimizing, i.e.

$$\lim_{n \to \infty} J(\theta_n) = J_* = \inf_{\theta \in X} J(\theta).$$

3) The sequence $\{\theta_n\} \subset \Lambda_0$ weakly converges to the point $\theta_{**} \in X_{**}$.

4) The following convergence rate is satisfied

$$0 \leq J(\theta_n) - J_* \leq \frac{c_1}{n}, \qquad c_1 = \text{const} > 0, \qquad n = 1, 2, \ldots$$

5) The boundary value problem (1.1)–(1.6) has a solution if and only if

$$\lim_{n \to \infty} J(\theta_n) = J_* = \inf_{\theta \in X} J(\theta) = J(\theta_{**}) = 0.$$

Proof. The first assertion follows from the fact that Λ_0 is bounded closed convex set of a reflexive Banach space X, as well as from the weak lower semi-continuity of functional $J(\theta)$ on weakly bicompact set Λ_0. The second assertion follows from estimation $J(\theta_n) - J(\theta_{n+1}) \geq \varepsilon \|\theta_n - \theta_{n+1}\|^2$, $n = 0, 1, 2, \ldots$ It follows that $J(\theta_{n+1}) < J(\theta_n)$, $\|\theta_n - \theta_{n+1}\| \to 0$ at $n \to \infty$, $\{\theta_n\} \subset \Lambda_0$. Hence from the convexity of functional $J(\theta_n)$ at Λ_0 follows, that $\{\theta_n\}$ is minimizing. The third assertion follows from weak bicompactness of set Λ_0. Estimation of convergence rate follows from inequality $J(\theta_n) - J(\theta_{**}) \leq c_1 \|\theta_n - \theta_{n+1}\|$. The last statement follows from Theorem 4.3. The theorem is proved. □

We note, that if $f(x,t)$, $f_{0j}(x,t)$, $j = \overline{1, m_2}$, $F(x,t)$ are linear functions with respect to x, then the functional $J(\theta)$ is convex.

Example

The equation of motion of the system is

$$\ddot{\varphi} + \varphi = \cos t, \qquad 0 < t < \pi, \qquad \varphi(0) = 0, \qquad \varphi(\pi) = 0, \tag{5.7}$$

where the phase constraint is given as

$$\frac{0.37t}{\pi} - 0{,}37 \leq \varphi(t) \leq \frac{0.44t}{\pi}, \qquad t \in I = [0, \pi]. \tag{5.8}$$

Integral constraint is defined by

$$\int_0^{\pi} \varphi(t)\,dt \le 1. \tag{5.9}$$

Denoting $\varphi = x_1,\ \dot\varphi = \dot x_1 = x_2$, the equation (5.7) can be written in vector form

$$\dot x = Ax + \overline{B}x + \mu(t), x(0) = x_0 = \begin{pmatrix} 0 \\ \delta \end{pmatrix} \in S_0, \qquad x(\pi) = \overline{x}_1 = \begin{pmatrix} 0 \\ \beta \end{pmatrix} \in S_1, \tag{5.10}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \overline{B} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad \mu(t) = \begin{pmatrix} 0 \\ \cos t \end{pmatrix},$$

$$x_0 \in S_0 = \{(x_1(0), x_2(0)) \in R^2 \mid x_1(0) = 0,\ x_2(0) = \delta \in R^1\},$$
$$x_1 \in S_1 = \{(x_1(\pi), x_2(\pi)) \in R^2 \mid x_1(\pi) = 0,\ x_2(\pi) = \beta \in R^1\},$$

phase constraint (5.8) has the form

$$\frac{0.37t}{\pi} - 0.37 \le x_1(t) \le \frac{0.44t}{\pi}, \qquad t \in I, \tag{5.11}$$

integral constraint (5.9) can be written as

$$\int_0^{\pi} x_1(t)\,dt \le 1. \tag{5.12}$$

For this task $F(x,t) = x_1$, $\gamma(t) = \frac{0.37t}{\pi} - 0.37$, $\delta(t) = \frac{0.44t}{\pi}$, $g_1(x_1) = \int_0^{\pi} f_{01}(x_1)\,dt = \int_0^{\pi} x_1(t)\,dt$, $c_1 = 1$, $m_1 = 1$, $m_2 = 0$, $f_{01} = x_1$.

Transformation

The function $\eta(t) = \eta_1(t)$, $t \in I$ where $\eta(t) = \int_0^t x_1(\tau)\,d\tau$, $\dot\eta(t) = x_1(t)$, $\eta(0) = 0$, $\eta(\pi) = 1 - d_1$, $d_1 \ge 0$.

The set $\Gamma = \{d_1 \in R^1 \mid d_1 \ge 0\}$. Let $\xi(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$, where $\xi_1(t) = x_1(t)$, $\xi_2(t) = x_2(t)$, $\xi_3(t) = \eta(t)$. Then

$$\dot\xi(t) = A_1\xi + \overline{B}_1\xi + \mu_1(t), \qquad t \in I = [0, \pi], \tag{5.13}$$

$$\xi(0) = \xi_0 = \begin{pmatrix} x_1(0) \\ x_2(0) \\ \eta(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix}, \qquad \xi(\pi) = \xi_1 = \begin{pmatrix} x_1(\pi) \\ x_2(\pi) \\ \eta(\pi) \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \\ 1 - d_1 \end{pmatrix}, \tag{5.14}$$

where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \overline{B}_1 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mu_1(t) = \begin{pmatrix} 0 \\ \cos t \\ 0 \end{pmatrix},$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad P\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \qquad B_2 = 0, \qquad \overline{f}_0 = 0.$$

The phase constraint can be written as

$$\frac{0.37t}{\pi} - 0.37 \le \xi_1(t) \le \frac{0.44t}{\pi}, \qquad t \in I = [0, \pi]. \tag{5.15}$$

Here $\delta \in R^1$, $\beta \in R^1$, $d_1 \in \Gamma = \{d_1 \in R^1 \mid d_1\} \ge 0$.

Immersion principle

Linear controlled system (see (4.1)–(4.3)) has the form

$$\dot{y} = A_1 y + B_1 w(t) + {}_1(t), \qquad t \in I = [0, \pi], \tag{5.16}$$

$$y(0) = \xi_0, \qquad y(\pi) = \xi_1, \qquad (\delta, \beta) \in R^2, \qquad w(\cdot) \in L_2(I, R^1),$$

where the matrix B_1 and linear homogeneous system $\dot{\omega} = A_1\omega$ are

$$B_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \qquad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \qquad \dot{\omega}_1 = \omega_2, \dot{\omega}_2 = 0, \dot{\omega}_3 = \omega_1.$$

The fundamental matrix of solution of the linear homogeneous system $\dot{\omega} = A_1\omega$ is determined by the formula

$$\theta(t) = e^{A_1 t} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ t & t^2/2 & 1 \end{pmatrix}, \quad \theta^{-1}(t) = e^{-A_1 t} = \begin{pmatrix} 1 & -t & 0 \\ 0 & 1 & 0 \\ -t & t^2/2 & 1 \end{pmatrix}, \quad \phi(t, \tau) = \theta(t)\theta^{-1}(\tau).$$

Vector

$$a = \Phi(0, \pi)\xi_1 - \xi_0 - \int_0^\pi \Phi(0, t)\mu_1(t)\, dt = \begin{pmatrix} -\pi\beta - 2 \\ \beta - \delta \\ \frac{\pi^2\beta}{2} + 1 - d_1 + \pi \end{pmatrix}.$$

Matrix

$$W(0, \pi) = \int_0^\pi \Phi(0, t) B B^* \Phi^*(0, t)\, dt = \begin{pmatrix} \pi^3/3 & -\pi^2/2 & -\pi^4/8 \\ -\pi^2/2 & \pi & \pi^3/6 \\ -\pi^4/8 & \pi^3/6 & \pi^5/20 \end{pmatrix}$$

$$W^{-1}(0, \pi) = \begin{pmatrix} 192/\pi^3 & 36/\pi^2 & 360/\pi^4 \\ 36/\pi^2 & 9/\pi & 60/\pi^3 \\ 360/\pi^4 & 60/\pi^3 & 720/\pi^5 \end{pmatrix}, \qquad W(0, t) = \begin{pmatrix} t^3/3 & -t^2/2 & -t^4/8 \\ -t^2/2 & t & t^3/6 \\ -t^4/8 & t^3/6 & t^5/20 \end{pmatrix},$$

$$W(t, \pi) = \begin{pmatrix} (\pi^3 - t^3)/3 & (t^2 - \pi^2)/2 & (t^4 - \pi^4)/8 \\ (t^2 - \pi^2)/2 & \pi - t & (\pi^3 - t^3)/6 \\ (t^4 - \pi^4)/8 & (\pi^3 - t^3)/6 & (\pi^5 - t^5)/20 \end{pmatrix}.$$

Then

$$\begin{aligned} \lambda_1(t, \xi_0, \xi_1) &= E(t)a = B_1^*(t)\Phi^*(0, t)W^{-1}(0, \pi)a \\ &= \frac{(-\pi\beta - 2)(-180t^2 + 192\pi t - 36\pi^2)}{\pi^4} + \frac{(\beta - \delta)[-30t^2 + 36\pi t - 9\pi^2]}{\pi^3} \\ &\quad + \left(\frac{\pi^2\beta}{2} + 1 - d_1 - \pi\right) \cdot \frac{-360t^2 + 360\pi t - 60\pi^2}{\pi^5}; \end{aligned}$$

$$\begin{aligned} E_1(t)\xi_0 &= \Phi(t, 0)W(t, \pi)W^{-1}(0, \pi)\xi_0 \\ &= \delta \begin{pmatrix} \frac{[12t(8\pi t - 3\pi^2 - 5t^2) + \pi t(\pi^3 + 18\pi t^2 - 9\pi^2 t - 10t^3)]}{\pi^3} \\ \frac{[\pi^3 + 18\pi t^2 - 9\pi^2 t - 10t^3]}{\pi^3} \\ \frac{[24t^2(8\pi t - 3\pi^2 - 5t^2) + \pi t^2(\pi^3 + 18\pi t^2 - 9\pi^2 t - 10t^3) + 3\pi t^3(3\pi t - \pi^2 - 2t^2)]}{2\pi^4} \end{pmatrix}, \end{aligned}$$

$$E_2(t)\xi_1 = \begin{pmatrix} \beta\cdot\frac{(-8\pi t^3+3\pi^2t^2+5t^4)}{2\pi^3} + (1-d_1)\cdot\frac{(-60\pi t^3+30\pi^2t^2+30t^4)}{\pi^5} \\ \beta\cdot\frac{(-12\pi t^2+3\pi^2t+10t^3)}{\pi^3} + (1-d_1)\cdot\frac{(-180\pi t^2+60\pi^2t+120t^3)}{\pi^5} \\ \beta\cdot\frac{(-2\pi t^4+\pi^2t^3+t^5)}{2\pi^3} + (1-d_1)\cdot\frac{(-15\pi t^4+10\pi^2t^2+6t^5)}{\pi^5} \end{pmatrix},$$

$$\mu_4(t) = \Phi(t,0)\int_0^t \Phi(0,\tau)\mu_2(\tau)d\tau - E_2(t)\int_0^\pi \Phi(\pi,t)\mu_2(t)dt = \begin{pmatrix} -\cos t + 1 + \frac{4\pi t^3-6\pi^2t^2}{\pi^4} \\ \sin t + \frac{12\pi t^2-12\pi^2t}{\pi^4} \\ t - \sin t + \frac{\pi t^4-2\pi^2t^2}{\pi^4} \end{pmatrix}.$$

Here

$$\begin{aligned} E(t) &= B_1^*(t)\phi^*(0,t)W^{-1}(0,\pi) \\ &= \Big(\frac{-180t^2+192\pi t-36\pi^2}{\pi^4},\ \frac{-30t^2+36\pi t-9\pi^2}{\pi^3},\ \frac{-360t^2+360\pi t-60\pi^2}{\pi^5}\Big), \\ E_2(t) &= \begin{pmatrix} \frac{28\pi t^3-12\pi^2t^2-15t^4}{\pi^4} & \frac{-8\pi t^3+3\pi^2t^2+5t^4}{2\pi^3} & \frac{-60\pi t^3+30\pi^2t^2+30t^4}{\pi^5} \\ \frac{84\pi t^2-24\pi^2t-60t^3}{\pi^4} & \frac{-12\pi t^2+3\pi^2t+10t^3}{\pi^3} & \frac{-180\pi t^2+60\pi^2t+120t^3}{\pi^5} \\ \frac{7\pi t^4-4\pi^2t^3-3t^5}{\pi^4} & \frac{-2\pi t^4+\pi^2t^3+t^5}{2\pi^3} & \frac{-15\pi t^4+10\pi^2t^2+6t^5}{\pi^5} \end{pmatrix}, \\ N_1(t) &= -E(t)\phi(0,\pi) \\ &= \Big(\frac{-180t^2+168\pi t-24\pi^2}{\pi^4},\ \frac{30t^2-24\pi t+3\pi^2}{\pi^3},\ \frac{360t^2-360\pi t+60\pi^2}{\pi^5}\Big), \\ N_2(t) &= -E_2(t). \end{aligned}$$

As follows from Theorem 4.1, the control

$$\begin{aligned} w(t) &= v(t) + \lambda_1(t,\xi_0,\xi_1) + N_1(t)z(t_1,v) \\ &= v(t) + \frac{(-\pi\beta-2)(-180t^2+192\pi t-36\pi^2)}{\pi^4} + \frac{(\beta-\delta)(-30t^2+36\pi t-9\pi^2)}{\pi^3} \\ &\quad + \Big(\frac{\pi^2\beta}{2}+1-d_1+\pi\Big)\frac{(-360t^2+360\pi t-60\pi^2)}{\pi^5} \\ &\quad + \frac{(-180t^2+168\pi t-24\pi^2)}{\pi^4}z_1(\pi,v) + \frac{(30t^2-24\pi t+3\pi^2)}{\pi^3}z_2(\pi,v) \\ &\quad + \frac{(360t^2-360\pi t+60\pi^2)}{\pi^5}z_3(\pi,v), \qquad t\in I=[0,\pi], \end{aligned} \tag{5.17}$$

where $z(t,v)$, $t\in I=[0,\pi]$ is solution of the differential equation

$$\dot z = A_1 z + B_1 v(t), \qquad z(0)=0, \qquad v(\cdot)\in L_2(I,R^1). \tag{5.18}$$

Solution of the differential equation (5.16) corresponding to equation (5.17) equals

$$\begin{aligned} y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} &= z(t,v) + \lambda_2(t,\xi_0,\xi_1) + N_2(t)z(\pi,v) \\ &= z(t,v) + E_1(t)\xi_0 + E_2(t)\xi_1 + \mu_4(t), \qquad t\in I=[0,\pi], \end{aligned}$$

where

$$
\begin{aligned}
y_1(t) = z_1(t,v) &+ \delta \frac{[12t(8\pi t - 3\pi^2 - 5t^2) + \pi t(\pi^3 + 18\pi t^2 - 9\pi^2 t - 10t^3)]}{\pi^3} \\
&+ \beta \frac{(-8\pi t^3 + 3\pi^2 t^2 + 5t^4)}{2\pi^3} + (1-d_1)\, \frac{(-60\pi t^3 + 30\pi^2 t^2 + 30t^4)}{\pi^5} \\
&+ \frac{4\pi t^3 - 6\pi^2 t^2}{\pi^4} - \cos t + 1 - \frac{(28\pi t^3 - 12\pi^2 t^2 - 15t^4)}{\pi^4} z_1(\pi, v) \\
&+ \frac{(8\pi t^3 - 3\pi^2 t^2 - 5t^4)}{2\pi^3} z_2(\pi, v) + \frac{60\pi t^3 - 30\pi^2 t^2 - 30t^4}{\pi^5} z_3(\pi, v),
\end{aligned}
\tag{5.19}
$$

$$
\begin{aligned}
y_2(t) = z_2(t,v) &+ \delta \frac{\pi^3 + 18\pi t^2 - 9\pi^2 t - 10t^3}{\pi^3} + \beta \frac{(-12\pi t^2 + 3\pi^2 t + 10t^3)}{\pi^3} \\
&+ (1-d_1) \frac{(-180\pi t^2 + 60\pi^2 t + 120t^3)}{\pi^5} + \sin t + \frac{12\pi t^2 - 12\pi^2 t}{\pi^4} \\
&- \frac{84\pi t^2 - 24\pi^2 t - 60t^3}{\pi^4} z_1(\pi, v) - \frac{(-12\pi t^2 + 3\pi^2 t + 10t^3)}{\pi^3} z_2(\pi, v) \\
&- \frac{(-180\pi t^2 + 60\pi^2 t + 120t^3)}{\pi^5} z_3(\pi, v),
\end{aligned}
\tag{5.20}
$$

$$
\begin{aligned}
y_3(t) &= z_3(t,v) \\
&+ \delta \frac{[24t^2(8\pi t - 3\pi^2 - 5t^2) + \pi t^2(\pi^3 + 18\pi t^2 - 9\pi^2 t - 10t^3) + 3\pi t^3(3\pi t - 2t^2 - \pi^2)]}{2\pi^4} \\
&+ \beta \frac{(-2\pi t^4 + \pi^2 t^3 + t^5)}{2\pi^3} + (1-d_1) \cdot \frac{(-15\pi t^4 + 10\pi^2 t^2 + 6t^5)}{\pi^5} + t - \sin t \\
&+ \frac{\pi t^4 - 2\pi^2 t^2}{\pi^4} - \frac{7\pi t^4 - 4\pi^2 t^3 - 3t^5}{\pi^4} z_1(\pi, v) - \frac{(-2\pi t^4 + \pi^2 t^3 + t^5)}{2\pi^3} z_2(\pi, v) \\
&- \frac{(-15\pi t^4 + 10\pi^2 t^2 + 6t^5)}{\pi^5} z_3(\pi, v), \qquad t \in I.
\end{aligned}
\tag{5.21}
$$

Note that $y_1(0) = 0$, $y_2(0) = \delta$, $y_3(0) = 0$, $y_1(\pi) = 0$, $y_2(\pi) = \beta$, $y_3(\pi) = 1 - d_1$.

Optimization problem

As for this example $f = y_1$, $F = y_1$, then optimization problem (4.12)–(4.15) can be written as: minimize the functional

$$
\begin{aligned}
J(v, p, d_1, \delta, \beta) &= \int_0^\pi \left[|w(t) - y_1(t)|^2 + |p(t) - y_1(t)|^2 \right] dt \\
&= \int_0^\pi F_0(t, v(t), p(t), d_1, \delta, \beta, z(t), z(\pi))\, dt \to \inf
\end{aligned}
\tag{5.22}
$$

at conditions (5.18), where

$$
v(\cdot) \in L_2(I, R^1), \qquad p(t) \in P(t), \qquad d_1 \in \Gamma, \qquad \delta \in R^1, \qquad \beta \in R^1, \tag{5.23}
$$

function $w(t)$, $t \in I$ is determined by the formula (5.17), and the function $y_1(t)$, $t \in I$ is defined by relation (5.19), set

$$P(t) = \left\{ p(\cdot) \in L_2(I, R^1) \mid \tfrac{0.37t}{\pi} - 0.37 \le p(t) \le \tfrac{0.44t}{\pi},\ t \in I \right\}. \tag{5.24}$$

The partial derivatives of F_0 read as follows:

$$\begin{aligned}
\frac{\partial F_0(t,q)}{\partial v} &= 2[w(t) - y_1(t)], \qquad \frac{\partial F_0(t,q)}{\partial p} = 2[p(t) - y_1(t)],\\
\frac{\partial F_0(t,q)}{\partial d_1} &= 2[w(t) - y_1(t)] \left(\frac{360t^2 - 360\pi t + 60\pi^2}{\pi^5} - \frac{60\pi t^3 - 30\pi^2 t^2 - 30t^4}{\pi^5} \right)\\
&\quad + 2[p(t) - y_1(t)] \left(\frac{-60\pi t^3 + 30\pi^2 t^2 + 30t^4}{\pi^5} \right);\\
\frac{\partial F_0(t,q)}{\partial \delta} &= 2[w(t) - y_1(t)] \left(\frac{-30t^2 + 36\pi t - 9\pi^2}{\pi^3} \right.\\
&\qquad \left. - \frac{[12t(8\pi t - 3\pi^2 - 5t^2) + \pi t(\pi^3 + 18\pi t - 9\pi^2 t - 10t^3)]}{\pi^3} \right)\\
&\quad + 2[p(t) - y_1(t)] \left[- \frac{[12t(8\pi t - 3\pi^2 - 5t^2) + \pi t(\pi^3 + 18\pi t^2 - 9\pi^2 t - 10t^3)]}{\pi^3} \right],\\
\frac{\partial F_0(t,q)}{\partial \beta} &= 2[w(t) - y_1(t)] \left[\frac{(-30t^2 + 36\pi t - 9\pi^2)}{\pi^3} - \frac{(-180t^2 + 180\pi t - 30\pi^2)}{\pi^3} \right.\\
&\qquad \left. - \frac{(-180t^2 + 192\pi t - 36\pi^2)}{\pi^3} - \frac{-8\pi t^3 + 3\pi^2 t^2 + 5t^4}{2\pi^3} \right]\\
&\quad + 2[p(t) - y_1(t)] \left(\frac{8\pi t^3 - 3\pi^2 t^2 - 5t^4}{2\pi^3} \right);\\
\frac{\partial F_0(t,q)}{\partial z_1} &= -2[w(t) - y_1(t)] - 2[p(t) - y_1(t)], \qquad \frac{\partial F_0(t,q)}{\partial z_2} = 0, \qquad \frac{\partial F_0(t,q)}{\partial z_3} = 0;\\
\frac{\partial F_0(t,q)}{\partial z_1(\pi)} &= 2[w(t) - y_1(t)] \cdot \left[\frac{-180t^2 + 168\pi t - 24\pi^2}{\pi^4} + \frac{28\pi t^3 - 12\pi^2 t^2 - 15t^4}{\pi^4} \right]\\
&\quad + 2[p(t) - y_1(t)] \cdot \left(\frac{28\pi t^3 - 12\pi^2 t^2 - 15t^4}{\pi^4} \right);\\
\frac{\partial F_0(t,q)}{\partial z_2(\pi)} &= 2[w(t) - y_1(t)] \left[\frac{30t^2 - 24\pi t + 3\pi^2}{\pi^3} - \frac{8\pi t^3 - 3\pi^2 t^2 - 5t^4}{2\pi^3} \right]\\
&\quad + 2[p(t) - y_1(t)] \left[- \frac{8\pi t^3 - 3\pi^2 t^2 - 5t^4}{2\pi^3} \right];\\
\frac{\partial F_0(t,q)}{\partial z_3(\pi)} &= 2[w(t) - y_1(t)] \left(\frac{360t^2 - 360\pi t + 60\pi^2}{\pi^5} - \frac{60\pi t^3 - 30\pi^2 t^2 - 30t^4}{\pi^5} \right)\\
&\quad + 2[p(t) - y_1(t)] \left(- \frac{60\pi t^3 - 30\pi^2 t^2 - 30t^4}{\pi^5} \right).
\end{aligned}$$

It is easy to verify that the functional (5.22) at conditions (5.18), (5.23), (5.24) is convex. Therefore, the sequences below are minimized.

Minimizing sequences

To this problem control $\theta = (v, p, d_1, \delta, \beta) \in X$. We choose the initial control

$$\theta_0 = (v_0(t), p_0(t), d_{10}, \delta_0, \beta_0) \in X,$$

where $v_0(\cdot) \in L_2(I, R^1)$, $p_0(t) \in P(t)$, $d_{10} \in \Gamma$, $\delta_0 \in R^1$, $\beta_0 \in R_1$. In particular $v_0(t) \equiv 1$, $p_0(t) = \frac{0.405t}{\pi} - 0.185 \in P(t)$, $d_{10} = 0.5$, $\delta_0 = -\frac{\pi}{8}$, $\beta_0 = -\frac{\pi}{8}$. We find the solution of the differential equation $\dot{z} = A_1 z + B_1 v(t)$, $z(0)=0$, at $v=v_0(t)$, i.e. $\dot{z}_{10} = \dot{z}_{20}$, $\dot{z}_{20} = -v_0(t)$, $\dot{z}_{30} = z_{10}$, $t \in [0, \pi]$, where $z_{10}(0) = 0$, $z_{20}(0) = 0$, $z_{30}(0) = 0$. Thus, $z_{10}(t), z_{20}(t), z_{30}(t)$, $t \in [0, \pi]$ are known. Then $q_0 = (v_0, p_0, d_1^0, \delta_0, \beta_0, z_0(t), z_0(\pi))$, $z_0(t) = (z_{10}(t), z_{20}(t), z_{30}(t))$. We compute the partial derivatives

$$\frac{\partial F_0(t, q_0)}{\partial v}, \quad \frac{\partial F_0(t, q_0)}{\partial p}, \quad \frac{\partial F_0(t, q_0)}{\partial d_1}, \quad \frac{\partial F_0(t, q_0)}{\partial \delta}, \quad \frac{\partial F_0(t, q_0)}{\partial \beta},$$

$$\frac{\partial F_0(t, q_0)}{\partial z_{10}}, \quad \frac{\partial F_0(t, q_0)}{\partial z_{20}} = 0, \quad \frac{\partial F_0(t, q_0)}{\partial z_{30}} = 0, \quad \frac{\partial F_0(t, q_0)}{\partial z_{10}(\pi)}, \quad \frac{\partial F_0(t, q_0)}{\partial z_{20}(\pi)}, \quad \frac{\partial F_0(t, q_0)}{\partial z_{30}(\pi)}.$$

Following approximations

$$v_1 = v_0 - \alpha_0 J_1'(\theta_0), \qquad p_1 = P_p[p_0 - \alpha_0 J_2'(\theta_0)], \qquad d_1 = P_\Gamma[d_{10}^0 - \alpha_0 J_3'(\theta_0)];$$

$$\delta_1 = \delta_0 - \alpha_0 J_4'(\alpha_0), \qquad \beta_1 = \beta_0 - \alpha_0 J_5'(\theta_0),$$

where

$$J_1'(\theta_0) = \frac{\partial F_0(t, q_0)}{\partial v} - B_1^* \psi_0(t) = \frac{\partial F_0(t, q_0)}{\partial v} + \psi_{20}(t), \qquad \psi_0 = (\psi_{10}, \psi_{20}, \psi_{30}),$$

$$J_2'(\theta_0) = \frac{\partial F_0(t, q_0)}{\partial p}, \qquad J_3'(\theta_0) = \int_0^\pi \frac{\partial F_0(t, q_0)}{\partial d_1}\, dt,$$

$$J_4'(\theta_0) = \int_0^\pi \frac{\partial F_0(t, q_0)}{\partial \delta}\, dt, \qquad J_5'(\theta_0) = \int_0^\pi \frac{\partial F_0(t, q_0)}{\partial \beta}\, dt.$$

Here $\psi_0(t) = (\psi_{10}(t), \psi_{20}(t), \psi_{30}(t))$, $t \in I$ is solution of the adjoint system

$$\dot{\psi}_{10} = \frac{\partial F_0(t, q_0)}{\partial z_{10}} - \psi_{20}, \qquad \dot{\psi}_{20} = \frac{\partial F_0(t, q_0)}{\partial z_{20}} = 0,$$

$$\dot{\psi}_{30} = \frac{\partial F_0(t, q_0)}{\partial z_{30}} - \psi_{10} = -\psi_{10},$$

$$\psi_{10}(\pi) = -\int_0^\pi \frac{\partial F_0(t, q_0)}{\partial z_{10}(\pi)} dt, \qquad \psi_{20}(\pi) = -\int_0^\pi \frac{\partial F_0(t, q_0)}{\partial z_{20}(\pi)} dt,$$

$$\psi_{30}(\pi) = -\int_0^\pi \frac{\partial F_0(t, q_0)}{\partial z_{30}(\pi)} dt.$$

The quantity $\alpha_0 = \frac{1}{K} = \text{const} = 0,1$. As a result, we find that $\theta_1 = (v_1, p_1, d_{11}, \delta_1, \beta_1)$. Further the process of constructing is repeated $\{\theta_n\}$ with the initial point θ_1, with the value

$\alpha_n = 0, 1$, $n = 0, 1, 2, \ldots$ It can be shown, that $v_n \to v_*$, $p_n \to p_*$, $d_{1n} \to d_{1*}$, $\delta_n \to \delta_*$, $\beta_n \to \beta_*$ at $n \to \infty$, the value $J = (v_*, p_*, d_{1*}, \delta_*, \beta_*) = 0$, where

$$v_*(t) = \frac{t}{2}\sin t - \frac{\pi}{4}\sin t, \qquad t \in [0, \pi];$$

$$p_*(t) = \frac{t}{2}\sin t - \frac{\pi}{4}\sin t, \qquad t \in [0, \pi], \qquad d_{1*} = 1, \qquad \delta_* = -\frac{\pi}{4}, \qquad \beta_* = -\frac{\pi}{4}.$$

Functions

$$z_1(v_*) = \cos t - 1 + \frac{1}{2}t\sin t - \frac{\pi}{4}\sin t + \frac{\pi}{4}t, \qquad t \in I,$$
$$z_2(v_*) = -\frac{1}{2}(\sin t - t\cos t) - \frac{\pi}{4}\cos t + \frac{\pi}{4}, \qquad t \in I,$$
$$z_3(v_*) = \sin t - t + \frac{1}{2}(\sin t - t\cos t) - \frac{\pi}{4}(-\cos t + 1) + \frac{\pi^2}{8}t^2, \qquad t \in I,$$

where $z_1(\pi, v_*) = \frac{\pi^2}{4} - 2$, $z_2(\pi, v_*) = 0$, $z_3(\pi, v_*) = \frac{\pi^3}{8} - \pi$.

Then (see (5.20)–(5.22))

$$y_{1*}(t) = \frac{1}{2}t\sin t - \frac{\pi}{4}\sin t = x_{1*}(t), \qquad t \in I;$$
$$y_{2*}(t) = x_{2*}(t) = -\frac{\pi}{4} + \frac{1}{2}\sin t + \frac{1}{2}t\cos t - \frac{\pi}{4}\cos t + \frac{\pi}{4}, \qquad t \in I;$$
$$y_{3*}(t) = \frac{1}{2}\sin t - \frac{1}{2}t\cos t + \frac{\pi}{4}\cos t - \frac{\pi}{4}, \qquad t \in I.$$

Solution of the initial boundary value problem (5.8)–(5.10):

$$\varphi(t) = x_{1*}(t) = y_{1*}(t), \quad t \in I, \qquad \frac{0.37t}{\pi} - 0.37 \le x_{1*}(t) \le \frac{0.44t}{\pi}, \quad t \in I;$$

$$\int_0^\pi \varphi(t)\,dt = \int_0^\pi x_{1*}(t)\,dt = 0 \le 1, \qquad x_0 = \left(0, -\frac{\pi}{4}\right), \qquad \overline{x}_1 = \left(0, -\frac{\pi}{4}\right),$$

$$\varphi(0) = 0, \qquad \dot{\varphi}(0) = -\frac{\pi}{4}, \qquad \varphi(\pi) = 0, \qquad \dot{\varphi}(\pi) = -\frac{\pi}{4}.$$

6 Conclusion

In general, the optimization problem (4.12)–(4.15) can have an infinite number of solutions $\{\theta_*\}$, for which $J(\{\theta_*\}) = 0$. Depending on the choice of the initial approximation the minimizing sequences converge to an element of the set $\{\theta_*\}$. Let $\theta_* = (v_1^*, v_2^*, \rho_*, d_*, x_0^*, x_1^*)$, where $J(\theta_*) = 0$ is a solution. Here $x_0^* = x(t_0), x_1^* = x(t_1), (x_0^*, x_1^*) \in S, x_0^*$ is the initial state of the system. The requirements imposed on the right-hand side of the differential equation (1.1) under which the initial Cauchy problem has a unique solution are represented in the formulation of the problem. Consequently, the differential equation (1.1) with the initial state $x(t_0 = x_0^*)$ has a unique smooth solution for the values $t \in [t_0, t_1]$. Moreover, $x(t_1) = x_1^*$ and all constraints (1.2)–(1.6) are satisfied. No matter what solution is allocated by iterative procedure, in the case of $J(\theta_*) = 0$ we find the appropriate solution to the problem (1.1)–(1.6).

For this example $x(0) = x_0^* = (0, -\frac{\pi}{4})$, $x(t_1) = x_1^* = (0, -\frac{\pi}{4})$. The differential equation (5.10), where $\dot{x} = (A + \bar{B})x + \mu(t)$, $x(0) = x_0^* = (0, -\frac{\pi}{4})$ has a unique solution

$$x_{1*}(t) = [2t\sin t - \pi\sin t]/4, \qquad x_{2*}(t) = [2\sin t + 2t\cos t - \pi\cos t]/4, \qquad t \in [0, \pi],$$

where $J(\theta_*) = 0$. Functions $x_{1*}(t),\ x_{2*}(t),\ t \in [0, \pi]$ satisfy the constraints (5.10)–(5.12), where $(x_{1*}(\pi), x_{2*}(\pi)) = x_1^*$.

The proof of the solvability and construction of solution of the boundary value problem based on solving of the optimization problem (4.12)–(4.15), where

$$\lim_{n\to\infty} J(\theta_n) = \inf_{\theta\in X} J(\theta) = 0$$

gives the solvability condition, and the solution of the boundary value problem is determined though the limit points of the sequence $\{\theta_n\}$ equal to $\{\theta_*\}$.

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